Fixed Points of the Conjugate in the Dual Steenrod Algebra

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What is a Conjecture?

A conjecture is a mathematical statement that is believed to be true but has not yet been proven.

Conjectures are often motivated by:

- observed patterns,
- computations,
- experimental evidence.

Why Study the Dual Steenrod Algebra?

- Algebraic Topology uses algebraic tools to understand shapes and surfaces
- The dual Steenrod algebra is a crucial algebraic tool

Abstract Algebra

A ring is a set equipped with multiplication and addition, satisfying a set of axioms.

Examples

The set of integers ($\mathbb{Z}$) and set of $n \times n$ matrices ($M_{n\times n}$) each form a ring. The set of natural numbers ($\mathbb{N}$) do NOT form a ring.

The Dual Steenrod Algebra

The dual Steenrod algebra is a polynomial ring over the field $\mathbb{F}_2$

$$A_* = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$$

where

- coefficients lie in $\mathbb{F}_2$,
- generators are $\xi_1, \xi_2, \ldots$

Finite Algebras H_n^j

To make computations more manageable, we study finite quotients of the dual Steenrod algebra.

$$H_n^j \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_j]/(\xi_i^{2^n} - \xi_i)$$

The parameters:

- j determines the number of generators,
- n determines the relation imposed on the generators.

Conjugation

The dual Steenrod algebra has additional structure as a Hopf algebra.

We focus on the conjugation map:

$$\chi : A_* \to A_*.$$

We write:

$$\overline{\xi_n} = \chi(\xi_n).$$

$$\chi(\xi_n) = \sum_{i=0}^{n-1} \xi_{n-i}^{2^i} \chi(\xi)$$

The conjugation map respects addition and multiplication:

$$\chi(a+b) = \chi(a) + \chi(b),$$

$$\chi(ab) = \chi(a)\chi(b).$$

Fixed Points and Conjugation Invariants

A fixed point (or conjugation invariant) is an element satisfying:

$$\chi(x) = x.$$

The project studies the conjugation invariants of finite algebras H_n^j.

The goal is to compute fixed points for small values of n and j and identify patterns that lead to conjectures.

Examples of Conjugation

The conjugation satisfies:

$$\chi(\xi_1) = \xi_1$$

so ξ_1 is a fixed point.

For the next generator:

$$\chi(\xi_2) = \xi_2 + \xi_1$$

so ξ_2 is not fixed.

However, combinations of elements can produce new fixed points.

Example: A Fixed Point in H_1^2

The conjugates are

$$\chi(\xi_2) = \xi_1 + \xi_2,$$

$$\chi(\xi_1\xi_2) = \xi_1 + \xi_1\xi_2.$$

Both contain the same extra term ξ_1.

Consider:

$$b = \xi_2 + \xi_1\xi_2.$$

Using $\chi(a+b) = \chi(a) + \chi(b)$,

$$\begin{aligned}\chi(b) &= (\xi_1 + \xi_2) + (\xi_1 + \xi_1\xi_2) \\ &= \xi_2 + \xi_1\xi_2 = b.\end{aligned}$$

Therefore, b is a fixed point.

Data

Fixed-Point Dimensions for $n = 1$

The number of basis elements, equivalently the dimension of $(H_1^j)^\chi$ over $\mathbb{F}_2$, is given below:

j	0	1	2	3	4	5	6	7	8	9	10
$\dim_{\mathbb{F}_2}\left((H_1^j)^\chi\right)$	1	2	3	6	10	20	36	72	136	272	528

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Conjecture

Let $(H_1^j)^\chi = \{x \in H_1^j \mid \chi(x) = x\}$ be the subspace fixed by the conjugation map χ. Then, for every integer $j \geq 0$,

$$\boxed{\dim_{\mathbb{F}_2}((H_1^j)^\chi) = \frac{2^j + 2^{\lceil j/2 \rceil}}{2}}.$$

Proposed Conjecture

$n\backslash j$	1	2	3	4	5	6	7	8	9	10
1	2	3	6	10	20	36	72	136	272	528
2	4	10	40	136	544	2,080	8,320	32,896	131,584	524,800
3	8	36	288	2,080	16,640	131,328	1,050,624	8,390,656	67,125,248	536,887,296
4	16	136	2,176	32,896	526,336	8,390,656	134,250,496	2,147,516,416	34,360,262,656	549,756,338,176
5	32	528	16,896	524,800	16,793,600	536,887,296	17,180,393,472	549,756,338,176	17,592,202,821,632	562,949,970,198,528

Conjecture

$$\dim_{\mathbb{F}_2}((H_j^n)^\chi) = \frac{2^{nj} + 2^{n\lceil j/2 \rceil}}{2}.$$

Strategies for Identifying the Fixed Points

This pattern appears many other places in nature.
Comparing to these may help us to analyze the fixed points.

Bit Strings up to Reversal

Define $[B_j]_\sim$ to be sets of equivalence classes of bit strings, such that $a \sim b$ if a and b are palindromes.

Examples

$[B_3]_\sim = [000], [100, 001], [010], [110, 011], [101], [111]$

Notice:

$$(H_1^3)^\chi = \mathbb{F}_2\{1, \xi_1, \xi_2\xi_1 + \xi_2, \xi_3\xi_2 + \xi_3, \xi_2 + \xi_3, \xi_2\xi_3 + \xi_1\xi_2\xi_3\}$$

$$\dim_{\mathbb{F}_2}(H_1^3)^\chi = \#[B_3]_\sim = 6.$$

Conjecture: Bijection Between Bits and $(H_1^j)^\chi$

We propose the following conjecture:

Conjecture

There exists a well defined bijection between elements of $(H_1^j)^\chi$ and $[B_j]_\sim$.

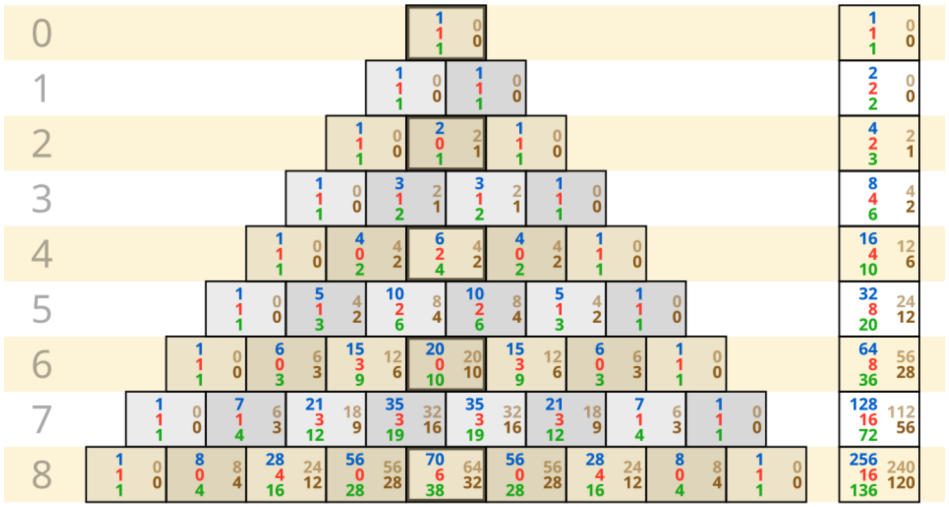
This would give us a formula for $(H_1^j)^\chi$.

Relation to Lozanić's triangle

Lozanić's triangle is a variant of Pascal's triangle deriving from the study of alkali.

It is a known fact that the sum of its rows is equal to $2^{n-2} + 2^{\lfloor n/2 \rfloor - 1}$.

Observation: The sums of the j'th row is equal to the dimension of $(H_1^{j+1})^\chi$.



Overview: Maps Between Three Structures

